**Bogoliubov Method**

**Bogoliubov approach**

Let’s go back to:



and we’ll assume V(q) exists in some sense. Often a δ function interaction is presumed, or we could argue for a T-matrix approximation like in previous file. In Bogoliubov’s approach, we presume we’re below Tc and so the gas has begun to collapse into the condensate. We further presume that N0, the occupation number for the k = 0 state (note both the condensate and normal liquid particles can occupy this state, though seems condensate prefers it more than the normal liquid, but is by no means restricted to it) is macroscopically large, on the order of N, and the occupation numbers of the rest of the k states are microscopic so that nk≠0 ~ 1 at best. Now from the excitation phenomenology file, we saw that in the limit T → 0, N0/N ~ 0.11, not 1, and further that the occupation numbers for the other k’s rather smoothly, not sharply, decreased in Gaussian fashion from the peak at k = 0; so there wasn’t a dramatic difference between nk=0 and nk=0.1 say. However, we know that when there is no interaction, i.e., when we’re dealing with a free gas making the transition into the Bose-Einstein condensate, we do have N0 ~ N, and nk≠0 ~ 1. So we might presume that in the limit of weak interaction, the Gaussian shape of nk might progressively approach a delta function, and so Bogoliubov’s assumption about N0 will get closer to being true.

Okay, well now consider the k = 0 creation/annihilation operators, and some many-body state with macrosocpic occupation of the k = 0 state, which I’ll just write as |N0> for short. Such states should include the ground state, as well as the low lying excited states. They will have the properties, as usual, that



The interesting thing is the approximation on the right. Usually N0 ~ 1. But in the condensate N0 ~ 1023. Since we end up with the same state both times, the approximations mean that c0†, c0 commute for all intents and purposes. Moreover, we can treat them as scalars, and replace each of the operators with their effect on the state, namely √N0. So what we’ll want to do is go back to H and see if we can do an expansion in powers of N0 basically, keeping just the largest terms. First the kinetic energy,



(recognizing ε0 = (0)2/2m = 0) and then the potential energy.



The 0 momentum terms split into four cases: k = 0, k´ = 0, k = -q, k´ = q. So we’ll separate out these terms,



and then,



There are a few more c0’s hiding. So,



Think we’re done now. And replacing the c0, c0† guys with √N0, as allowed since that is their action on any state (in the k eigenbasis):



The N02 term is just a constant. We’ll keep the N0 terms, and neglect the √N0 terms and the constant term as these should be much smaller as soon as we’re in the condensate. So then,



where n0 = N/V. We can ignore the first term in PE because it just gives n0NeV2(0), which is just a constant. We’re discarding information about how many particles are in the condensate though, so I wonder if we’d be able to get N0, Ne self-consistently by doing this. Whatever. Altogether then,



We could assume V2(**q**) is even in q, ‘cause:



And so I imagine we could simplify this expression somewhat by strategically changing the variable of summation from k → -k. And would arrive at:



I guess I’ll stick with the first though, since that’s what Mahan does. So this can be solved exactly since it’s quadratic. Could do via GF’s and finding the self-energy. Could do like we did when we solved the Mean Field Superconductor Hamilltonian (in the Appendix). Let’s try a different way, by finding the operator which annihilates excitations. We’ll presume it’s some linear combination of all the different operators in H. Something like:



where α, α´, β, β´ are to be determined. Note this equation would hold for all k, including negative k. We’re not separating k into positive/negative values or anything. Then it should have the property that:



So let’s try it. First I’ll write H as,



Then, keeping the sum over k implicit, to save space,



So first we have, using [nk,cq] = -cqδkq, and [nk, c†q] = c†qδkq,



and then, using [AB,C] = A[B,C] + [A,C]B, we have:



And employing the δ function in the implicit sum over k, we have:



now grouping terms by operator, we have:



Demanding this equals -EqAq gives us four equations:



Maybe there’s a faster way. But now we have a 4×4 matrix equation to diagonalize. But looks like it separates into two separate 2×2 matrices. The α, β´ guys are self-consistent, as are the α´, β guys. So we need to solve:



We can find the eigenvalues of each 2×2 matrix separately. So look at the bottom one,



I’ll take V´q = V´-q, and then ε´q = ε´-q as well. Then,



So we have:



Might note that this spectrum was also found for the case of particles interacting via a δ function in 1D. If we had used the top matrix equation, we can see we’d have obtained the same result. What are the eigenvectors? Well it seems there could be four different ones, since our overall matrix is 4D. Of course our matrix isn’t Hermitian, so it might be that there aren’t four separate ones? Well, I suppose we can also rule out the negative energy solution, and so that means there could only be two separate eigenvectors. Well I’ll start with the bottom guy again,



We can eliminate one of the rows (let’s do bottom one) due to linear dependence, and then have our un-normalized eigenvector would be:



Using,



This simplifies to:



And so the normalized eigenvector becomes:



So we have the αq and β´-q coefficients of Aq. And it seems that people just set the α´-q and βq coefficients to zero? This is logically possible as it satisfies the equations. So we say:



and so we can say (sans perhaps a constant I’m neglecting),



We still have a problem though, because V2(k) isn’t defined. We have to replace V(k) with T(k). He basically argues that V(k) should be a constant, i.e., that we can use a delta function approximation to V(r). And we choose its strength so that it gives the same phase shift to the relative wavefunction in the two-body problem as the exact potential does. We end up with:



where a is some scattering length. This makes the energies come out to:



In the low q limit, we’d have:



This is an acoustic sound wave spectrum, with velocity v = m-1√(4πan0). So we succesfully predicted acoustic vibrations at low T – just as is shown in neutron scattering experiments. And at large k it turns into the free particle spectrum, as can see, and as we kind of see from the large q part of the experimental spectrum. So those two qualitative features of the excitation spectrum are reproduced here 😊. Bogoliubov spectrum looks like this on left, vs. the experimental one on right.

Chart, line chart

Description automatically generated

If we consider the Lennard-Jones potential to be a hard sphere repulsion with range a = σ0 introduced before as the ‘range’ of the Lennard – Jones potential, then we find a speed of sound equal to 130m/s, ‘close’ to the actual value of 220m/s. Unfortunately, Mahan says, the agreement is rather coincidental. The Lennard – Jones potential is attractive at long distances, and this can make *a* take on different values, or even negative ones – indicating bound states. A more careful analysis makes the discrepancy even worse ☹. So apparently we cannot take the 4He liquid to be weakly interacting.

In the low q limit, since εq ~ q2, and Eq ~ q, the annihilation operator takes the form,



and accordingly,



Excitations would then take the form,



Would be nice to see what an elementary excitation looks like in position space. So consider:



where |0> is the vacuum. But this overlap will give us zero, since we’ll have the overlap between states with different particle numbers. But to our calculations accuracy, we’ll note that Aq† is proportional to the density operator, since:



So let’s make this replacement, sans the multiplicative factor:



Now fill in what cq is (see 2nd Quantization notes in QM/Identical Particles Folder).



So,



Now we need to commute the ψ†(r) all the way to the left, where it will annihilate the |0>. First two commutations give us:



Continuing the process, we can see we’ll end up with:



Last term is zero, and so we’re left with:



And this is just:



which is a result we’ll see again in a second. This is basically a density wave added on top of the ground state. This describes harmonic oscillations of the ‘lattice’ of He atoms, and is kind of what we’d expect in an excitation wavefunction, since we know that the small q excitations have an acoustic phonon-like dispersion curve. Interestingly, in the opposite limit, high q, we see that Eq and εq are roughly the same, and so the annihilation operator approaches:



So this indicates the excitations become particle-like in this limit, which does accord with the fact that Eq → εq. So in summary, I guess low q (momentum) excitations only exist in the form of acoustic waves. At higher q, the interatomic interaction becomes less and less able to support such rapid frequency oscillations. So the collective oscillations break up and the particles start moving independently – basically acting as free particles. And so the energy spectrum turns into the free particle spectrum. But we still have a problem, in that the dip in the energy spectrum is unaccounted for. We can account for that in the next file.